International Journal of Theoretical Physics, Vol. 37, No. 7, 1998

Clebsch–Gordan Coefficient for *q,s*-Deformed Two-Dimensional Hydrogen Atom

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Received September 30, 1997

Utilizing the $SU(2)_{q,s}$ symmetry of the *q,s*-deformed two-dimensional hydrogen atom (2DHA), the Clebsch–Gordan coefficient for the *q,s*-deformed 2DHA is derived in the Bargmann space.

INTRODUCTION

The study of the hydrogen atom has been a tantalizing problem to scientists for years (Schrödinger, 1926, 1967; Bhaumik *et al.*, 1986; Nandy *et al.*, 1989; Nauenberg, 1989; Gay *et al.*, 1989; Lena *et al.*, 1991; Gerry and Kiefer, 1988; Prunele, 1990). Utilizing the $SU(2)_{q,s}$ symmetry of the *q,s*-deformed two-dimensional hydrogen atom (2DHA), we calculate the Clebsch–Gordan coefficient for the *q, s*-2DHA. First we introduce the Bargmann representations in the tensor product space of the irreducible representations for $SU(2)_{q,s}$ and then derive the Bargmann expressions for the bases of irreps, the coherent state, and the operators. Finally, the Clebsch–Gordan coefficient is easily obtained.

1. $SU(2)_{q,s}$ SYMMETRY OF *q,s*-DEFORMED 2DHA

The Hamiltonian for a general 2DHA in the center-of-mass frame is

$$H = \frac{\bar{p}^2}{2\mu} - \frac{e^2}{r} \tag{1}$$

where μ is the reduced mass, $r = \sqrt{x^2 + y^2}$, and $\bar{p} = p_x\bar{i} + p_y\bar{j}$.

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0020-7748/98/0700-204 3$15.00/0

We first define a two-dimensional Lenz vector

$$\vec{m} = \frac{1}{2\mu e^2}(\vec{p} \times \vec{l} - \vec{l} \times \vec{p}) - \vec{e}_r \tag{2}$$

where $\vec{e}_r = (x\vec{i} + y\vec{j})/r$ is the radial unit vector and $\vec{l} = (xp_y - yp_x)\vec{k} = l_z\vec{k}$ is the angular momentum. We have the commutation relations

$$[H, l_z] = [H, \vec{m}] = 0 \tag{3}$$

and the confinement conditions

$$\vec{m} \cdot \vec{l} = \vec{l} \cdot \vec{m} \tag{4}$$

If we choose natural units such that $\hbar = 1$, we get easily

$$[l_z, m_\pm] = \pm m_\pm, \qquad [m_+, m_-] = \frac{-4\mathscr{H}}{\mu e^2} l_z \tag{5}$$

where $m_\pm = m_x \pm i m_y$. Equation (5) denotes that operators l_z and $m_\pm$ cannot construct a closed Lie algebra. In order to overcome this difficulty, it is necessary to restrict the energy eigenvalues of the 2DHA to a subspace spanned by all the degeneracy states with energy range $E < 0$. Making the transformations

$$J_z = l_z, \qquad J_+ = \sqrt{\frac{-\mu e^2}{2E}}\, m_+, \qquad J_- = \sqrt{\frac{-\mu e^2}{2E}}\, m_- \tag{6}$$

we have

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z \tag{7}$$

Therefore we have a closed Lie algebra.

In consideration of the equivalence between a 2DHA and a two-dimensional isotropic harmonic oscillator with a confinement condition, say, the 2DHA has the $SU(2)_{q,s}$ symmetry (Xu, 1991). Here we can naturally generalize this 2DHA to the q,s-deformed case, namely, considering the q,s-deformation of the two-dimensional isotropic harmonic oscillator with a confinement condition. Correspondingly, (7) becomes (Jing, 1993)

$$[J'_z, J'_\pm] = \pm J'_\pm, \qquad s^{-1}J'_+J'_- - sJ'_-J'_+ = s^{-2J'_z}[2J'_z] \tag{8}$$

where the q-bracket is defined as $[x] = (q^x - q^{-x})/(q - q^{-1})$. For future convenience, we also define the q,s-bracket $[x]_{q,s} = s^{1-x}[x]$.

2. COHERENT STATE OF THE *q,s*-DEFORMED 2DHA AND THE BARGMANN REPRESENTATIONS

The unitary irreducible representations of $SU(2)_{q,\ s}$ can be represented by the quantum number j (denoted $T^{(j)}$, where $j = 0, 1/2, 1, \ldots$). The action of the $SU(2)_{q,s}$ generators on the bases $|j, m\rangle'$ is given by

$$J'_+ |j, m\rangle' = \sqrt{[j - m]_{q,s^{-1}}[j + m + 1]_{q,s}}\, |j, m + 1\rangle' \tag{9}$$

$$J'_- |j, m\rangle' = \sqrt{[j + m]_{q,s}[j - m + 1]_{q,s^{-1}}}\, |j, m - 1\rangle' \tag{10}$$

$$J'_Z |j, m\rangle' = m|j, m\rangle' \tag{11}$$

The coherent state of the $T^{(j)}$ irreducible representation for $SU(2)_{q,s}$ is written as

$$|jz) = e^{z^* J'_-}_{q,s,s^{-1}}|j, j\rangle' = \sum_{m=-j}^{j} \sqrt{\frac{[2j]_{q,\ s}!}{[j + m]_{q,s}!\ [j - m]_{q,\ s}!}}\,(z^*)^{j-m}|j, m\rangle' \tag{12}$$

with the following definition for the deformed exponential:

$$e^{x}_{q,s,s^{-1}} = \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{[n]_{q,s}!\ [n]_{q,s^{-1}}!}} \tag{13}$$

The normalization coefficient is

$$A_j(|z|^2) = \sum_{m=-j}^{j} \frac{[2j]_{q,s}!}{[j + m]_{q,s}!\ [j - m]_{q,s}!}\,(|z|^2)^{j-m} \tag{14}$$

In order to construct the completeness relation of the quantum state $|jz)$, we define $P_{q,s,s^{-1}}(j - m, z)$ to be an observable probability of $|j, m\rangle'$ in the state $|jz)$, namely

$$P_{q,s,s^{-1}}(j - m, z) = |{}'\langle j, m|jz)|^2 = \frac{[2j]_{q,s}!}{[j + m]_{q,s}!\ [j - m]_{q,s}!}\,(|z|^2)^{j-m} \tag{15}$$

Letting

$$P_{q,s,s^{-1}}(j - m) = \int P_{q,s,s^{-1}}(j - m, z)\, dz^2$$

and with ρ representing the density matrix of the state $|j, m\rangle'$, we have

$$\rho = \sum_{m=-j}^{j} P_{q,s,s^{-1}}(j-m)|j, m\rangle'\,'\langle j, m|$$

Thus, the completeness relation of $|jz)$ can be written as

$$\frac{1}{\pi}\rho^{-1}\int \frac{|jz)(jz|}{A_j(|z|^2)}\,dz^2 = 1 \tag{16}$$

Now we define the Bargmann representation of the bases $|j, m\rangle'$ for the $T^{(j)}$ irreducible representation as follows:

$$f_{jm}(z) = (jz^*|j, m\rangle' = \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m} \tag{17}$$

Defining a state vector in the space of the $T^{(j)}$ irreducible representation $|\psi\rangle' = \sum_{m=-j}^{i} C_m|j, m\rangle'$, we have

$$(jz^*|J'_+|\psi\rangle' = \sum_m C_m(jz^*|J'_+|j, m\rangle'$$

$$= \sum_m C_m[j-m]\sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m-1} \tag{18}$$

We also have

$$\frac{1}{z^*}\left[z^*\frac{d}{dz^*}\right](jz^*|\psi\rangle'$$

$$= \sum_m C_m\,[j-m]\sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m-1} \tag{19}$$

From equations (18) and (19), we get the Bargmann representation of operator J'_+,

$$B_j(J'_+) = \frac{1}{z^*}\left[z^*\frac{d}{dz^*}\right] \tag{20}$$

Similarly, we have

$$(jz^*|J'_-|\psi\rangle' = \sum_m C_m(jz^*|J'_-|j, m\rangle'$$

$$= \sum_m C_m[j+m]_{q,s}\,s^{j-m}\sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m+1} \tag{21}$$

$$z^* s^{z^* d/dz^*}\left[2j - z^* \frac{d}{dz^*}\right]_{q,s} (jz^*|\psi\rangle'$$
$$= \sum_m C_m[j+m]_{q,s} s^{j-m} \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m+1} \tag{22}$$

$$(jz^*|J'_z|\psi\rangle' = \sum_m C_m(jz^*|J'_z|\psi\rangle'$$
$$= \sum_m C_m m \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m} \tag{23}$$

$$(j - z^{*d/dz^*})\,(jz^*|\psi\rangle'$$
$$= \sum_m C_m m \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}}\,(z^*)^{j-m} \tag{24}$$

so we can obtain the Bargmann representations of operators of J'_- and J'_z,

$$B_j(J'_-) = z^* s^{z^* d/dz^*}\left[2j - z^* \frac{d}{dz^*}\right]_{q,s}, \qquad B_j(J'_z) = j - z^* \frac{d}{dz^*} \tag{25}$$

3. THE TENSOR PRODUCT OF IRREDUCIBLE REPRESENTATIONS

Noticing the operator structures in the reducible space $T^{(j)}$,

$$J'^{(j)}_{\pm} = J'^{(j_1)}_{\pm} \otimes (s^{-1}q)^{J'_z(j_2)} + (sq)^{-J'_z(j_1)} \otimes J'^{(j_2)}_{\pm} \tag{26}$$

$$J'^{(j)}_z = J'^{(j_1)}_z \otimes I^{(j_2)} + I^{(j_1)} \otimes J'^{(j_2)}_z \tag{27}$$

where $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$, we can obtain the Bargmann representations of operators J'_z and $J'_\pm$, respectively:

$$B_j(J'_\pm) = B_{j_1}(J'_\pm) \otimes (s^{-1}q)^{B_{j_2}(J'_z)} + (sq)^{-B_{j_1}(J'_z)} \otimes B_{j_2}(J'_\pm) \tag{28}$$

$$B_j(J'_z) = B_{j_1}(J'_z) \otimes B_{j_2}(I) + B_{j_1}(I) \otimes B_{j_2}(J'_z) \tag{29}$$

Accordingly to the Baker–Campbell–Hausdorff formula $e^A B e^{-A} = B + [A, B] + (1/2!)\,[A, [A, B]] + \cdots$, we can obtain

$$q^{-nJ'_z} J'_- = q^n J'_- q^{-nJ'_z}, \qquad J'_- q^{nJ'_z} = q^n q^{nJ'_z} J'_-$$

Accordingly, we have

$$(J_-'^{(j)})^n = \{J_-'^{(j_1)} \otimes (s^{-1}q)^{J_z'(j_2)} + (sq)^{-J_z'(j_1)} \otimes J_-'^{(j_2)}\}^n$$
$$= \sum_{l=0}^{n} \frac{[n]_{q,s}!}{[l]_{q,s}!\,[n-l]_{q,s}!} (sq)^{l(n-l)} \{J_-'^{(j_1)} \otimes (s^{-1}q)^{J_z'(j_2)}\}^l$$
$$\times \{(sq)^{-J_z'(j_1)} \otimes J_-'^{(j_2)}\}^{n-l} \tag{30}$$

4. THE BARGMANN REPRESENTATIONS FOR IRREDUCIBLE BASES IN TENSOR PRODUCT SPACE

First we solve the Bargmann representation of the irreducible bases $|j, j\rangle'$ in the reducible process $T^{(j_1)} \otimes T^{(j_2)} = \Sigma \oplus T^{(j)}$: $f_{j,j}(z_1, z_2) = (j_1 z_1^*, j_2 z_2^* | j, j\rangle'$, where $|j_1 z_1^*, j_2 z_2^*) = |j_1 z_1^*) \otimes | j_2 z_2^*)$. For $f_{j,j}(z_1, z_2)$ we have

$$B_j(J_z'^{\otimes}) f_{j,j}(z_1, z_2) = j\, f_{j,j}(z_1, z_2) \tag{31}$$

$$B_j(J_+'^{\otimes}) f_{j,j}(z_1, z_2) = 0 \tag{32}$$

where the sign $\otimes$ in (31) and (32) denotes that the active space is the tensor product space. From (20) and (25)–(27) the Bargmann representations are given by

$$B_j(J_z'^{\otimes}) = \left(j_1 - z_1^* \frac{d}{dz_1^*}\right) + \left(j_2 - z_2^* \frac{d}{dz_2^*}\right) \tag{33}$$

$$B_j(J_+'^{\otimes}) = \frac{1}{z_1^*}\left[z_1^* \frac{d}{dz_1^*}\right] \otimes (s^{-1}q)^{j_2 - z_2^* d/dz_2^*} + (sq)^{z_1^* d/dz_1^* - j_1} \otimes \frac{1}{z_2^*}\left[z_2^* \frac{d}{dz_2^*}\right] \tag{34}$$

Assume that the eigenvalue of (31) is $z_1^{*n} z_2^{*m}$; then we have

$$B_j(J_z'^{\otimes})(z_1^{*n} z_2^{*m}) = (j_1 - n) z_1^{*n} z_2^{*m} + (j_2 - m) z_1^{*n} z_2^{*m} = j z_1^{*n} z_2^{*m} \tag{35}$$

so that $n + m = j_1 + j_2 - j$.

We can introduce a new parameter k, and let $k = n + m = j_1 + j_2 - j$, where $k = 0, 1, 2, \ldots, 2\min(j_1, j_2)$. Accordingly, $f_{j,j}(z_1, z_2)$ can be written as

$$f_{j,j}(z_1, z_2) = \sum_{n=0}^{k} C_n\, z_1^{*n} z_2^{*k-n} \tag{36}$$

Substituting (36) into (32), we have

$$\sum_{n=1}^{k} C_n[n](s^{-1}q)^{j_2-k+n} z_1^{*n-1} z_2^{*k-n} + \sum_{n=0}^{k-1} C_n[k-n](sq)^{n-j_1} z_1^{*n} z_2^{*k-n-1}$$

$$= \sum_{n=0}^{k-1} \{C_{n+1}[n+1](s^{-1}q)^{j_2-k+n+1} + C_n[k-n](sq)^{n-j_1}\} z_1^{*n} z_2^{*k-n-1} = 0 \quad (37)$$

where the recurrence formula is given by

$$C_{n+1} = -C_n \frac{[k-n]}{[n+1]} s^{2n+j_2-j_1-k+1} q^{k-j_1-j_2-1} \quad (38)$$

Therefore

$$C_n = (-1)^n \frac{[k]!}{[n]!\,[k-n]!} s^{n(j-2j_1)+n^2-1} q^{-n(j+1)} \quad (39)$$

so that $f_{j,j}(z_1, z_2)$ can be rewritten as

$$f_{j,j}(z_1, z_2) = C \sum_{n=0}^{k} \frac{(-1)^n}{[n]!\,[k-n]!} s^{n(j-2j_1)+n^2-1} q^{-n(j+1)} z_1^{*n} z_2^{*k-n} \quad (40)$$

where C is the normalization coefficient(the constant $[k]_{q,s^{-1}}!$ is implied in C), and can be determined from

$$(f_{j,j}, f_{j,j}) = C^*C \sum_{n=0}^{k} \sum_{m=0}^{k} \frac{(-1)^{n+m}}{[n]!\,[m]!\,[k-n]!\,[k-m]!}$$
$$\times\, s^{(m+n)(j-2j_1)+n^2+m^2-2} q^{-(m+n)(j+1)}$$
$$\times \frac{1}{\pi^2} \rho_1^{-1} \rho_2^{-1} \int\int \frac{z_1^n z_1^{*m} z_2^{k-n} z_2^{*k-m}}{A_{j_1}(|z_1|^2) A_{j_2}(|z_2|^2)} dz_1^2\, dz_2^2 = 1 \quad (41)$$

where

$$\rho_1 = \sum_{m_1=-j_1}^{j_1} P_{q,s,s^{-1}}(j_1 - m_1) |j_1, m_1\rangle' \, '\langle j_1, m_1| \quad (42)$$

$$\rho_2 = \sum_{m_2=-j_2}^{j_2} P_{q,s,s^{-1}}(j_2 - m_2) |j_2, m_2\rangle' \, '\langle j_2, m_2| \quad (43)$$

For convenience, letting C be a real number, we can obtain easily

$$C^2 = [2j_1]_{q,s}!\,[2j_2]_{q,s}!/\Omega \quad (44)$$

with

$$\Omega = \sum_{n=0}^{k} s^{2n(j-2j_1)+2(n^2-1)} q^{-2n(j+1)} \frac{[2j_1 - n]_{q,s}![2j_2 - k + n]_{q,s}!}{[n]_{q,s}!\,[k-n]_{q,s}!}$$

In the following discussion, we solve the Bargmann representations of $f_{j,m}(z_1, z_2)$. From (10) we have

$$(J'_-)^{j-m}\big|j, j\rangle' = \sqrt{\frac{[2j]_{q,s}!\,[j-m]_{q,s^{-1}}!}{[j+m]_{q,s}!}}\,\big|j, m\rangle' \tag{45}$$

Then

$$\begin{aligned} f_{j,m}(z_1, z_2) &= (j_1\, z_1^*, j_2\, z_2^*\,\big|j, m>' \\ &= \sqrt{\frac{[j+m]_{q,s}!}{[2j]_{q,s}!\,[j-m]_{q,s^{-1}}!}}\,(j_1\, z_1^*, j_2 z_2^*\,\big|(J'_-)^{j-m}\big|\,j, j>' \\ &= \sqrt{\frac{[j+m]_{q,s}!}{[2j]_{q,s}!\,[j-m]_{q,s^{-1}}!}}\,B_j((J'^{\otimes}_-)^{j-m})f_{j,j}(z_1, z_2) \end{aligned} \tag{46}$$

where $B_j((J'^{\otimes}_-)^{j-m})$ can be determined by equations (25) and (30), i.e.,

$$\begin{aligned} B_j\left((J'^{\otimes}_-)^{j-m}\right) &= \sum_{l=0}^{j-m} \frac{[j-m]_{q,s}!}{[l]_{q,s}!\,[j-m-l]_{q,s}!}\,(sq)^{l(j-m-l)} \\ &\times \left\{\left(z_1^* s^{z_1^* d/dz_1^*}\,[2j_1 - z_1^*\, d/dz_1^*]_{q,s}\right)^l (sq)^{(l-j+m)(j_1 - z_1^* d/dz_1^*)}\right\} \\ &\otimes \left\{(s^{-1}q)^{l(j_2 - z_2^* d/dz_2^*)}(z_2^* s^{z_2^* d/dz_2^*}\,[2j_2 - z_2^*\frac{d}{dz_2^*}]_{q,s})^{j-m-l}\right\} \end{aligned} \tag{47}$$

Substituting equations (40) and (47) into equation (46), we have

$$\begin{aligned} f_{j,m}(z_1,z_2) &= C s^{(j-m)(j_2-j)-1} q^{j_1(m-j)}[j-m]_{q,s}!\sqrt{\frac{[j+m]_{q,s}!}{[2j]_{q,s}![j-m]_{q,s^{-1}}!}} \\ &\times \sum_{l=0}^{j-m}\sum_{n=0}^{j_1+j_2-j} \frac{(-1)^n([2j_1-n]_{q,s})^l([j_2-j_1+j+n]_{q,s})^{j-m-l}}{[n]![j_1+j_2-j-n]![l]_{q,s}![j-m-l]_{q,s}!} \\ &\times s^{l(j_1+j-j_2-m-l)+n(j-2j_1)} q^{l(2j-m-l)-n(m+1)}(z_1^*)^{l+n}(z_2^*)^{j_1+j_2-m-l-n} \end{aligned} \tag{48}$$

5. CLEBSCH–GORDAN COEFFICIENTS

From (12), we have

$$
\begin{aligned}
(j_1 z_1^*, j_2 z_2^* | jz) &= \sum_{m=-j}^{j} \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}![j-m]_{q,s}!}} (z^*)^{j-m} (j_1 z_1^*, j_2 z_2^* | j, m\rangle' \\
&= \sum_{m=-j}^{j} \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}!\,[j-m]_{q,s}!}} (z^*)^{j-m} f_{j,\,m}(z_1, z_2) \qquad (49)
\end{aligned}
$$

Substituting (48) into (49) we have

$$
\begin{aligned}
(j_1 z_1^*, j_2\, z_2^* | jz) = {} & C s^{j(j_2-j)-1}\, q^{-j_1 j} \\
& \times \sum_{m=-j}^{j} \sum_{l=0}^{j-m} \sum_{n=0}^{j_1+j_2-j} s^{\,l(j_1+j-j_2-m-l)+n(j-2j_1)+m(j-j_2)} \\
& \times q^{\,l(2j-m-l)-n(m+1)+mj_1} \sqrt{[j-m]_{q,s}! \,/\, [j-m]_{q,s^{-1}}!} \\
& \times \frac{(-1)^n\, ([2j_1-n]_{q,s})^l\, ([j_2-j_1+j+n]_{q,s})^{j-m-l}}{[n]!\,[j_1+j_2-j-n]!\,[l]_{q,s}!\,[j-m-l]_{q,s}!} \\
& \times (z_1^*)^{l+n}\, (z_2^*)^{j_1+j_2-m-l-n}\, (z^*)^{j-m} \qquad (50)
\end{aligned}
$$

In order to calculate the Clebsch–Gordan coefficient, assume that $j_1 - m_1 = n + l$; then $\Sigma_l\, \Sigma_n$ can be replaced by $\sum_{m_1=-j_1}^{j_1} \sum_{n=0}^{j_1+j_2-j}$ and equation (50) is rewritten as

$$
\begin{aligned}
& (j_1 z_1^*, j_2 z_2^* | jz) \\
& = C \sum_{m=-j}^{j} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \delta_{m-m_1,m2} \\
& \quad \times s^{j(j_2-j)-1+(j_1-m_1)(j-j_2-m_2)+m(j-j_2)}\, q^{j_1(m-j)+(j_1-m_1)(2j-j_1-m_2)} \\
& \quad \times \sqrt{[j-m]_{q,s}! / [j-m]_{q,s^{-1}}!} \\
& \quad \times \left\{ \sum_{n=0}^{j_1+j_2-j} \frac{(-1)^n ([2j_1-n]_{q,s})^{j-m_1-n} ([j_2-j_1+j+n]_{q,s})^{j-j_1-m_2+n}}{[n]!\,[j_1+j_2-j-n]!\,[j_1-m_1-n]_{q,s}!\,[j-j_1-m_2+n]_{q,s}!} \right. \\
& \quad \left. \times s^{n(j_2-j_1+m_2-m_1-n)}\, q^{n(2j_1-2j-2m_1-n-1)} \right\} \times (z^*)^{j-m} (z_1^*)^{j_1-m_1} (z_2^*)^{j_2-m_2} \qquad (51)
\end{aligned}
$$

On the other hand, we have

$$(j_1 z_1^*, j_2 z_2^* | jz) = \sum_{m=-j}^{j} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} (j_1 z_1^*, j_2 z_2^* | j_1 m_1, j_2 m_2\rangle' \times {}'\langle j_1 m_1, j_2 m_2 | j,m\rangle' \, {}'\langle j, m | jz) \tag{52}$$

where ${}'\langle j_1 m_1, j_2 m_2 | j, m\rangle'$ is the Clebsch–Gordan coefficient. For other factors, we have

$$(j_1 z_1^*, j_2 z_2^* | j_1 m_1, j_2 m_2\rangle' = (j_1 z_1^* | j_1 m_1\rangle' \times (j_2 z_2^* | j_2 m_2\rangle'$$

$$= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sqrt{\frac{[2j_1]_{q,s}! \, [2j_2]_{q,s}!}{[j_1+m_1]_{q,s}! \, [j_1-m_1]_{q,s}! \, [j_2+m_2]_{q,s}! \, [j_2-m_2]_{q,s}!}} \times (z_1^*)^{j_1-m_1} (z_2^*)^{j_2-m_2} \tag{53}$$

$${}'\langle j,m | jz) = ((jz^* | j,m\rangle')^* = \sqrt{\frac{[2j]_{q,s}!}{[j+m]_{q,s}! \, [j-m]_{q,s}!}} (z^*)^{j-m} \tag{54}$$

Substituting equations (51), (53), and (54) into (52), we find the Clebsch–Gordan coefficient

$${}'\langle j_1 m_1, j_2 m_2 | j, m\rangle' = \delta_{m-m_1,m_2} A^{-1} s^{j(j_2-j)-1+(j_1-m_1)(j-j_2-m_2)+m(j-j_2)} \times q^{j_1(m-j)+(j_1-m_1)(2j-j_1-m_2)} [j-m]_{q,s}!$$

$$\times \sqrt{\frac{[j+m]_{q,s}! \, [j_1+m_1]_{q,s}! \, [j_1-m_1]_{q,s}! \, [j_2+m_2]_{q,s}! \, [j_2-m_2]_{q,s}!}{[2j]_{q,s}! \, [j-m]_{q,s^{-1}}!}}$$

$$\times \sum_{n=0}^{j_1+j_2-j} \frac{(-1)^n ([2j_1-n]_{q,s})^{j_1-m_1-n} ([j_2-j_1+j+n]_{q,s})^{j-j_1-m_2+n}}{[n]! \, [j_1+j_2-j-n]! \, [j_1-m_1-n]_{q,s}! \, [j-j_1-m_2+n]_{q,s}!}$$

$$\times s^{n(j_2-j_1+m_2-m_1-n)} q^{n(2j_1-2j-2m_1-n-1)} \tag{55}$$

$$A^2 = [2j_1]_{q,s}! \, [2j_2]_{q,s}! / ([j_1+j_2-j]_{q,s} C)^2 \tag{56}$$

comes from equation (44).

REFERENCES

Bhaumik, D., *et al.* (1986). *Journal of Physics A*, **19**, 1353.
Gay, J. C., *et al.* (1989). *Physics Review A*, **39**, 6587.
Gerry, C., and Kiefer, J. (1988). *Physics Review A*, **37**, 665.
Jing, S. C. (1993). *Modern Physics Letters A*, **8**, 543.

Lena, C., *et al.* (1991). *Europhysics Letters*, **15**, 697.
Nandy, S., *et al.* (1989). *Journal of Physics A*, **22**, 1005.
Nauenberg, M. (1989). *Physics Review A*, **40**, 1133.
Prunele, E. D. (1990). *Physics Review A*, **42**, 2542.
Schrödinger, E. (1926). *Naturwissenschaften*, **14**, 664.
Schrödinger, E. (1967). *On Letters in Wave Mechanics*, Philosophical Library, New York.
Xu, B. W. (1991). *Journal of Nature*, **14**, 553 [in Chinese].